An Entropic Approach to Analyze Investor Utility Involving a Financial Structured Product

Sukanto Bhattacharya and Kuldeep Kumar*

We propose an entropic model of extrinsic utility arising out of the element of choice regarding portfolio re-balancing strategies available to an individual investor who has chosen to invest in a financial structured product with a terminal payoff same as that from a rainbow option. We also propose a generalization of our posited framework by incorporating a fuzzy measure of probabilistic uncertainty concerning the nature of the structured financial product.

Keywords: financial structured product, rainbow option, tracking portfolio, investor utility, Shannon entropy, Markov process, fuzzy measure

JEL classification: C44, C65, G11

1 Introduction

In the parlance of applied finance, a self-financing portfolio is a portfolio of multiple financial assets which does not require any further injection of funds beyond its inception. Such portfolios are usually constructed to track the performance of financial structured products or benchmarks and require periodic re-balancing to minimize the tracking error, i.e., the gap between the portfolio performance and performance of the specific product or benchmark being tracked. Usually the re-balancing activity involves a re-direction of funds from the underperforming assets to the assets that are observed performing to potential (Fabozzi, 1998).

Suppose a financial structured product is made up of a basket of $n$ different assets such that the investor has the right to claim the value of the best-performing asset out of that basket after a stipulated holding period. Now if one of the $n$ assets

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*Authors are respectively at: Business Administration Department, Alaska Pacific University, U.S.A. and Faculty of Business, Bond University, Australia. We are grateful to the referees for comments on the earlier version of the paper.
is risk-free and all the others are risky, then the investor is assured of a minimum return equal to the risk-free rate \( r_f \) on his or her invested capital at the termination of the stipulated holding period. This effectively means that his or her investment becomes endogenously capital-guaranteed as the terminal wealth, even in the worst-case scenario, cannot be lower than the initial wealth invested at the risk-free rate minus a finite cost of portfolio insurance.

Such a financial structured product would invariably be in essence a rainbow option with a typical terminal payoff function as follows (Johnson, 1987):

\[
\text{Payoff}_T = \max \left[ \max_j (XT), V_0 e^{rT} \right], j = 1, 2, \ldots, n-1
\]

In the above equation, \((XT)_j\) is the terminal value of asset \( X_j \), \( V_0 \) is the initial wealth, and \( T \) is the length of the holding period. Extending the delta-hedging argument to a multi-asset scenario would mean that a tracking portfolio would consist of investments in the \( n-1 \) underlying risky assets plus the risk-free asset.

The proportion of funds to be invested at time \( t \) in a risky asset \( X_j \) is given as follows:

\[
(K_t)_j = \frac{\frac{\partial V_t}{\partial (X_j)}}{\sum_j \frac{\partial V_t}{\partial (X_j)}}
\]

Here \((K_t)_j\) is the proportion of funds to be invested at time \( t \) in the risky asset \( X_j \) and \( V_t \) is the value of the rainbow option at time \( t \). There will, however, always be a finite magnitude of tracking error due to practical considerations of time and because transaction costs rule out continuous re-balancing, which is an underlying assumption in the Black-Scholes option valuation model from which the delta-hedging strategy originates.

A number of papers have been published on possible computational algorithms to minimize the tracking error, possibly involving rather elaborate dynamic programming or control-theoretic modeling schemes (e.g., Bhattacharya and Khoshnevisan, 2005). The object is to keep the tracking problem essentially forecast-free so as not to compound the tracking error by introducing additional forecasting errors. The usual approaches primarily involve a strategy to optimize the trade-off between transaction costs generated by frequent re-balancing activity and
portfolio insurance costs generated by the tracking error (Leland and Rubinstein, 1998).

2 Information coding form of the portfolio re-balancing choices

Very little theoretical research has gone into the exploration of the utility forms underlying the investor choices involving the self-financing tracking portfolio. Computational economists of late have been actively seeking to formulate an entropic measure of utility by applying the mathematical paradigms of classical information theory (Abbas, 2002). In this paper we attempt to carry this line of thinking into the analysis of utility considerations underlying investments in financial structured products by transforming the investor choice set with respect to the tracking portfolio re-balancing decision into an information coding form.

For a tracking portfolio \( P = \{X_1, X_2, \ldots, X_n\} \) with \( n - 1 \) underlying risky assets plus one risk-free asset, one can envisage a binary response from the investor in terms of the re-balancing decision whereby the investor either takes funds out of an underlying asset or puts more funds into it at each and every re-balancing point. A financial structured product with \( n - 1 \) risky assets would have \( 2^{n-1} \) elements in the investor choice set and would be considered a \( 2^{n-1} \)-bit product.

3 Model formulation

Actually the information-theoretic measure of investor utility underlying the re-balancing decision of a multi-asset tracking portfolio can be tied to the general Markovian property of asset prices that underlies the Black-Scholes option-valuation model, which fundamentally assumes that asset prices evolve according to a stochastic diffusion process of geometric Brownian motion. If \( V \) stands for the value of an asset (or the value of a portfolio of several assets), the Brownian motion model makes the following assumptions:

1. \( V_0 = 0 \).
2. \( V_t - V_s \) is a random variable that is normally distributed with mean zero and variance \( t - s \).
3. $V_t - V_s$ is independent of $V_y - V_z$ if $(s, t)$ and $(y, z)$ are non-overlapping time intervals.

The third property implies that the Brownian motion is a Markovian process with no long-term memory. The switching behavior of asset prices from “high” (bull state) to “low” (bear state) and vice versa according to a Markovian state-transition rule constitutes a well-researched topic in stochastic finance. In fact, it has been proven that steady-state equilibrium exists when the state probabilities are equalized for a stationary transition probability matrix (Bhattacharya and Samanta, 2003).

The steady-state equilibrium corresponds to the condition of strong efficiency in the financial markets whereby no historical market information, public or private, can result in consistent arbitrage opportunities. In terms of a tracking portfolio for a $2^{m-1}$-bit financial structured product based on the payoff of a rainbow option, the optimal (i.e., utility-maximizing) re-balancing decision will be ultimately influenced by the Markovian behavior of the prices of underlying assets given by the $n \times n$ transition probability matrix presented in Table 1.

Table 1: An $n \times n$ transition probability matrix for assets within a financial structured product

<table>
<thead>
<tr>
<th>Best performing asset at $t+1$</th>
<th>Asset 1</th>
<th>Asset 2</th>
<th>Asset 3</th>
<th>...</th>
<th>Asset n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Best performing asset at $t$</td>
<td>Asset 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P(1</td>
<td>1)$</td>
<td>$P(2</td>
<td>1)$</td>
<td>$P(3</td>
</tr>
<tr>
<td></td>
<td>Asset 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P(1</td>
<td>2)$</td>
<td>$P(2</td>
<td>2)$</td>
<td>$P(3</td>
</tr>
<tr>
<td></td>
<td>Asset 3</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$P(1</td>
<td>3)$</td>
<td>$P(2</td>
<td>3)$</td>
<td>$P(3</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>Asset n</td>
<td>$P(1</td>
<td>n)$</td>
<td>$P(2</td>
<td>n)$</td>
<td>$P(3</td>
</tr>
</tbody>
</table>

Analogously, as in mathematical information theory, a similar Markovian structure is employed to improve the encoding of a source alphabet. For each state in the Markov system, an appropriate code can be obtained from the corresponding transition probabilities of leaving that state. The efficiency gain depends on how variable (temporally unstable) the probabilities are for each state. However, as the order of the Markov process is increased, the efficiency gain tends to be less and less as the number of attainable states approaches infinity (Wolfowitz, 1957).

The strength of the Markov formulation lies in its capacity to handle correlation between successive states. If $S_1, S_2, \ldots, S_m$ are the first $m$ states of a stochastic
variable, what is the probability that the next state will be \( S_i \)? This is written as the conditional probability \( P(S_i \mid S_1, S_2, \ldots, S_n) \). Then the Shannon measure of information from a state \( S_i \) is given as follows (Shannon, 1948):

\[
I(S_i \mid S_1, S_2, \ldots, S_n) = \log_2 \left[ P(S_i \mid S_1, S_2, \ldots, S_n) \right]^{-1}
\]

Following the canonical definition of entropy as expected information, the entropy of the Markov process is then expressible as follows (Hamming, 1986):

\[
H(S_i) = \sum_j P(S_j \mid S_1, S_2, \ldots, S_n) I(S_i \mid S_1, S_2, \ldots, S_n)
\]

Now coming back to our 2\(^{-n}\)-bit financial structured product, at each re-balancing point \( t \), one of the \( n \) assets in the self-financing tracking portfolio will turn out to be the “best performer,” say \( X_{it}^* \). Therefore we are interested in the entropy of the Markov process where at the \( i \) th re-balancing point the \( j \) th asset within the tracking portfolio turns out to be the best performer, i.e., \((X_j)_{i,t} = X_{it}^* \).

Exporting the entropy formulation for an \( m \)-state Markov process to our self-financing tracking portfolio corresponding to a 2\(^{-n}\)-bit financial structured product, the extrinsic utility of the \( j \) th asset at the \( i \) th re-balancing point is therefore obtained as follows:

\[
U(X_j)_i = \sum \left\{ P(X_{i+1}^*, X_{i+2}^*, \ldots, X_{i+n}^* \mid (X_j)_{i,t} = X_{i,t}^*, X_{i+1}^*, X_{i+2}^*, \ldots, X_{i+n}^*) \right\}
\]

Here the entropic extrinsic utility measure \( U(X_j)_i \) refers to the utility of choice in terms of available re-balancing strategies made available to the investor by the presence of the \( j \) th asset within the tracking portfolio at the \( i \) th re-balancing point. Thus, the utility of an asset within a self-financing tracking portfolio corresponding to a 2\(^{-n}\)-bit financial structured product goes beyond the intrinsic risk-return characteristics of that asset—it carries an extra utility dimension (as quantified by the entropic measure) by virtue of providing an additional element of choice to the investor in terms of available re-balancing options for the tracking portfolio.
4 Numerical illustration

Here we have considered actual historical market data to construct a three-asset, capital-guaranteed financial structured product. The three risky assets are gold futures, the Lehman Brothers Growth Fund, and a well-diversified portfolio of equities proxied by the S&P500 index. The return on the risk-free asset is proxied by the 3-month T-bill rate. The periodicity of the data is monthly and the span is two years—from February 2, 2001, to February 3, 2003. A notional amount of US$100,000 is invested on February 2, 2001, in a dynamically managed structured product with a floor of $90,000. The risk-free rate is an annualized 2.40% determined as the average observed rate on 3-month T-bills for the holding period. The asset return correlations (assumed stationary) are provided in the correlation matrix shown in Table 2.

Table 2: \(3 \times 3\) asset returns correlation matrix for gold futures, LB Growth Fund, and index portfolio

<table>
<thead>
<tr>
<th>Correlation Matrix</th>
<th>S&amp;P500 Index</th>
<th>Lehman Brothers Growth Fund</th>
<th>Gold Futures</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P500 Index</td>
<td>1</td>
<td>0.8629</td>
<td>-0.1946</td>
</tr>
<tr>
<td>Lehman Brothers Growth Fund</td>
<td>0.8629</td>
<td>1</td>
<td>-0.3470</td>
</tr>
<tr>
<td>Gold Futures</td>
<td>-0.1946</td>
<td>-0.3470</td>
<td>1</td>
</tr>
</tbody>
</table>

The overall market during the period under consideration was strongly bearish with the average monthly return on S&P 500 index a miserable −1.61% and that on the Lehman Brothers Growth Fund was not much better at −1.49%. Gold futures performed reasonably, well generating a monthly average return of 1.20%.

Therefore the product we have constructed is an 8-bit financial structured product consisting of three risky assets and one risk-free asset. We employ a mechanism of dynamic portfolio management termed constant proportion portfolio insurance (CPPI), which exposes a constant multiple of a cushion over an investor’s floor or “insured” value to the performance of the risky asset. For example, an investor with a $100 million portfolio, a floor of $90 million and a constant multiple of 5 will allocate $50 million (i.e., \(5 \times 100 - 90\)) million to the risky asset and the remaining $50 million to the risk-free asset. The investor will re-balance the exposures as the portfolio value changes.
Unlike other portfolio insurance strategies, CPPI does not require the investor to specify a finite investment horizon. In our CPPI scheme, the constant multiple is 2 and the floor is $90,000. The terminal portfolio value on February 3, 2003, is seen to be approximately $100,061, i.e., the capital guarantee mechanism endogenous to our 8-bit structured product has just about worked, ensuring that the value of the portfolio at the end of the horizon did not go below the initial capital invested. Though the holding period return is lower in this case (only about 0.461%) compared to the return on the best-performing asset (i.e., gold futures), the main benefit of the CPPI approach is that with a constant multiple $k > 0$, the investor obtains a higher participation in the market component and therefore, is afforded an opportunity to reap a proportionately higher reward if and when one of the risky assets within the portfolio starts to perform really well over a period of time.

In the actual market data we used, the ideal investment would have been to invest everything in gold futures, which would have generated a holding period return of about 33%. However this return cannot be obtained realistically as it calls for perfect foresight. But the nature of this financial structured product guarantees the investor an assured return over the investment horizon anywhere roughly between 0.33% and 33% with zero downside risk.

The rational objective is obviously to get as close as possible to the upper limit of the return range. The investor’s choice set (related to the number of assets within the envelope) then becomes an ideal vehicle for cardinalizing utility. It may be argued that a five-year time horizon instead of a two-year one would have provided a more comprehensive picture, but then again we have treated the utility emanating from an effective capital-guarantee mechanism as our primary objective and it was primarily with respect to that objective that an overwhelmingly bearish period was chosen to measure the extrinsic utility of the portfolio insurance scheme when the capital guarantee mechanism is triggered.

In our numerical example, if we constructed a 4-bit product out of the 8-bit one by randomly leaving out one of the three risky assets, we would still have a capital guaranteed structure. But there will then be a higher probability that now the maximum return the investor could get is no longer 33% but only the 5.06% approximate return obtainable by investing all of the $100,000 in the risk-free asset for the period. This is because there is a probability of 1/3 that the best performer
(i.e. gold futures) could be left out. This demonstrates that, in general, higher bit structured products will offer greater utility to the investor by expanding their choice set thereby broadening the return range. The trade-off between the increased overheads of having additional assets and the benefits of an expanded opportunity set is something open to future exploration.

In a financial structured product with three underlying risky assets plus one risk-free asset, the best performer may be hypothesized to be traceable by a first-order Markov process whereby the best performing asset at time $t+1$ depends on the best performing asset at time $t$. In our numerical illustration, we obtain the state-transition matrix reported in Table 3.

<table>
<thead>
<tr>
<th>Transition Matrix</th>
<th>S&amp;P500</th>
<th>Lehman Bros.</th>
<th>GC00</th>
<th>Risk-free rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P500</td>
<td>0.0000</td>
<td>1.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>Lehman Bros.</td>
<td>0.1429</td>
<td>0.2857</td>
<td>0.4286</td>
<td>0.1429</td>
</tr>
<tr>
<td>GC00</td>
<td>0.0000</td>
<td>0.1818</td>
<td>0.6364</td>
<td>0.1818</td>
</tr>
<tr>
<td>Risk-free rate</td>
<td>0.0000</td>
<td>0.2000</td>
<td>0.4000</td>
<td>0.4000</td>
</tr>
</tbody>
</table>

Then for a one-step Markov process, the extrinsic utilities offered to the investor by each of the assets included in this structured product can be computed as follows using the information-theoretic measure of extrinsic utility:

$$U(S&P500) = \sum P(S&P500) I(S&P500 \mid Lehman \text{ Bros., GC00}, r_f)$$

$$= 0.0416$$

$$U(\text{Lehman Bros.}) = \sum P(\text{Lehman Bros.}) I(\text{Lehman Bros.} \mid S&P500, GC00, r_f)$$

$$= 0.250$$

$$U(GC00) = \sum P(GC00) I(GC00 \mid S&P500, \text{Lehman Bros.}, r_f)$$

$$= 0.500$$

$$U(r_f) = \sum P(r_f) I(r_f \mid S&P500, \text{Lehman Bros., GC00})$$

$$= 0.208$$

What is especially interesting to note is that the Lehman Brothers Growth Fund actually has a higher extrinsic utility as measured by our entropic formula than the risk-free rate although on an average the risk-free rate outperformed the Lehman
Brothers Growth Fund over the two-year investment period. This is due to the fact that though intrinsic utility of the risk-free rate was higher than that of the Lehman Brothers Growth Fund, the latter provided a greater element of choice to the investor, primarily due to a substantially wider range of volatility, such that it ended up as the best performer more often compared to the risk-free asset even though, on an average, it was easily outperformed by the risk-free rate.

5 Extending to a fuzzy measure

Now consider a variable $V$ whose exact value, which lies in the space denoted $N = \{\Phi_1, \Phi_2, \ldots, \Phi_n\}$, is not completely known. In many such situations, the best one can do is to formulate his or her knowledge about $V$ by means of a fuzzy measure on $N$. A situation like this can be envisaged with the selection of assets underlying the financial structured product we have been discussing. Even when one of the underlying assets is theoretically risk free, thereby making such a financial structured product endogenously capital-guaranteed, there would most likely still be a substantial variety in the types of underlying assets which investors with varying utility profiles would prefer. This is because of the widely varying intrinsic risk-return characteristics of different classes of financial assets. The available choices to the investor given his or her utility preferences determine the universe of discourse. The more uncertain are an investor’s utility preferences, the wider is the range of available choices and the greater is the degree of fuzziness involved in determining the true nature of the tracking portfolio (Allen et al., 2003).

For each subset $S \subseteq N$ of values, let $\mu(S)$ represent a fuzzy measure associated with one’s belief (or self-confidence) that the value of $V$ is contained in the subset $S$. An assumption of sufficient monotonicity of $\mu$ implies that one cannot be less confident that $V$ lies in a smaller set than a larger one. Further, we assume for sake of simplicity that the fuzzy measure is additive, i.e., for any $(S, T) \subseteq N$ one has $\mu(S + T) = \mu(S) + \mu(T)$. Then the average marginal contribution of the element $\Phi_j$ to a combination not containing the same is given by the Shapley index $\theta_j(\mu)$, which is the $i$th component of the Shapley value $\theta(\mu)$ given below:

$$\theta(\mu) = \{\theta_1(\mu), \theta_2(\mu), \ldots, \theta_n(\mu)\} \in [0,1]^n$$
Given our assumptions of monotonicity and additivity of $\mu$, the following result is obtained through the standard procedures of fuzzy set theory:

$$H_i(\mu) = H_u(\mu) = H(\omega) \text{ and } \omega_i = \mu(\{\Phi_i\}), \; i = 1, 2, \ldots, n$$

Here $H_i(\mu)$ and $H_u(\mu)$ are the generalized lower and upper Shannon entropies of the fuzzy measure $\mu$ on $N$ (Marichal and Roubens, 1999). In terms of our financial structured product, when there is an element of fuzziness associated with the contents of the tracking portfolio, it necessarily implies that the value of the asset which is the best performer at re-balancing point $t$ also becomes fuzzy, i.e., $X^*_t$ is not exactly known but it lies in the space $P = \{X_1, X_2, \ldots, X_n\}$. Then the Shapley index $\theta_i(\mu)$ may be interpreted as a measure of the average confidence boost an investor may obtain by adding the asset $X_j$ to some sub-portfolio (a subset of the true tracking portfolio) not originally containing it.

Basically, the additivity assumption associated with the fuzzy measure $\mu$ makes one’s uncertainty about the true nature of the tracking portfolio with respect to the inclusion/non-inclusion of asset $X_j$ a probabilistic uncertainty rather than a qualitative uncertainty. In this context, therefore, the Shannon entropy of this probability distribution can serve as an adequate measure of the uncertainty concerning the inclusion/non-inclusion of $X_j$ in the tracking portfolio.

6 Conclusion

There is some uniformity in views among economists as well as physicists that a functional correspondence exists between the formalisms of economic theory and classical thermodynamics. The laws of thermodynamics can be intuitively interpreted in an economic context, and the correspondences show that thermodynamic entropy and economic utility are related concepts sharing comparable formal framework. Utility is said to arise from that component of thermodynamic entropy whose change in value is due to irreversible transformations (Foley, 1994). In this paper however we have not resorted to the standard Carnot entropy given by $dS = \tau Q/T$ where $S$ is the entropy measure, $Q$ is the thermal energy of the state transformation (irreversible), and $T$ is absolute temperature (Smith, 1998). Rather, we have employed the Shannon entropy which, although it
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has its philosophical basis in the thermodynamic form, has a somewhat different mathematical representation. In the process we believe to have posited a promising framework for achieving a seamless integration between a facet of applied financial economics and classical information theory.

We have further proposed, in an attempted generalization of our framework, that a fuzzy measure may be appropriate to computationally capture one’s uncertainty about the true nature of the financial structured product with respect to the inclusion/non-inclusion of a particular asset $X_i$ within the self-financing tracking portfolio from among those available for consideration. Perhaps a natural next step would be to consider a computational algorithm that seeks to implement our posited entropic modeling methodology of investor utility in a real-world investment scenario, ideally accommodating a fuzzy logic module as well to account for cases where the investor’s preference set is fuzzy.

References


